Bras and kets

Basic notions

1. $\underbrace{\langle f|}_{\text{"bra"}}\underbrace{g\rangle}_{\text{"ket"}}$ binary (scalar) product — a bracket

A ket-bra:

$$|g\rangle\langle f|$$

is an operator:

$$|g\rangle\underbrace{\langle f|h\rangle}_{c\text{ (number)}} = c\cdot|g\rangle$$

which maps $|h\rangle$ (any h non orthogonal to $\langle f|$) into $c\cdot|g\rangle$.

$\langle f|\ldots$ wants to form a scalar product to the right

$\ldots|g\rangle$ wants to form a scalar product to the left.

2. Projectors

Let $\{|i\rangle\}$ be an orthonormal basis $(i = 1, 2, \dots \infty)$ so that

$$\langle i|k\rangle = \delta_{ik} = \begin{cases} 0 & i \neq k \\ 1 & i = k. \end{cases}$$

Then

$$\hat{P}_i = |i\rangle\langle i|$$

is an idempotent operator:

$$\hat{P}_i^2 = |i\rangle\underbrace{\langle i|i\rangle}_{1}\langle i| = |i\rangle\langle i| = \hat{P}_i$$

and it projects any vector (nonorthogonal to $\langle i|$) to $|i\rangle$:

$$\hat{P}_i|v\rangle = |i\rangle\underbrace{\langle i|v\rangle}_{\text{number } c} = c \cdot |i\rangle$$

$$\hat{P}_{ik} = |i\rangle\langle i| + |k\rangle\langle k|$$

is also idempotent, and projects to the 2D space spanned by $|i\rangle$, $|k\rangle$:

$$\hat{P}_{ik}|v\rangle = |i\rangle\underbrace{\langle i|v\rangle}_{c_i} + |k\rangle\underbrace{\langle k|v\rangle}_{c_k}$$

$$= c_i|i\rangle + c_k|k\rangle$$

etc.

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3. Resolution of identity

$$\hat{I} = \sum_i^{(all)} |i\rangle\langle i| \Longleftarrow \boxed{\text{Resolution of identity}}$$

projects to all $|i\rangle$, that is does not project: it is the operator of unity.

Proof: $|v\rangle = \sum_j v_j |j\rangle$

$$\hat{I}|v\rangle = \sum_{ij} |i\rangle\langle i| v_j |j\rangle$$

$$= \sum_{ij} v_j |i\rangle\underbrace{\langle i|j\rangle}_{\delta_{ij}} = \sum_i v_i|i\rangle$$

$$= |v\rangle$$

Qed.

4. Spectral resolution

Let

$$\hat{H}|i\rangle = E_i |i\rangle$$

Here $|i\rangle$ is no longer an arbitrary orthonormal basis, but now they are the eigenvectors of the Hermitian operator $\hat{H}$.

Then $\hat{H}$ can be represented as:

$$\hat{H} = \sum_j E_j |j\rangle\langle j|$$

(spectral resolution of $\hat{H}$)

Proof:

$$\hat{H}|i\rangle = \sum_j E_j |j\rangle \underbrace{\langle j|i\rangle}_{\delta_{ji}}$$

$$= E_i |i\rangle \quad \text{, q.e.d.}$$

5. Operator functions

5.a Powers:

$$\hat{H}^2 = \sum_{ik} E_i E_k |i\rangle \underbrace{\langle i|k\rangle}_{\delta_{ik}} \langle k|$$

$$= \sum_i E_i^2 |i\rangle\langle i|$$

or

$$\hat{H}^n = \sum_i E_i^n |i\rangle\langle i|$$

5.b.

- $$e^{\hat{H}} = \sum_i e^{E_i} |i\rangle\langle i|$$

- $$\hat{H}^{-1} = \sum_i \frac{1}{E_i} |i\rangle\langle i|$$

Proof:

$$\hat{H}^{-1}\hat{H} = \sum_{ik} \frac{E_k}{E_i} |i\rangle \underbrace{\langle i|k\rangle}_{\delta_{ik}} \langle k|$$

$$= \sum_i |i\rangle\langle i| = \hat{I}, \quad \text{q.e.d.}$$

- for any Taylor-expandable function $f(\hat{H})$

$$f(\hat{H}) = \sum_i f(E_i) |i\rangle\langle i|$$