Elements of the variational calculus

1. Variation of a function

$$f(x) \longrightarrow f(x) + \underbrace{\delta f}_{\text{a small change in } f}$$

2. Definition of a functional

$f \to \mathcal{F}(f) =$ a number, which depends on $f(x)$

Example:

$$f \to \int_a^b f(x)\,dx \quad , \text{ etc.}$$

3. Variation of a functional

$$\underset{\substack{\uparrow \\ \text{an infinitesimal number}}}{\delta \mathcal{F}} = \underbrace{\mathcal{F}(f+\delta f) - \mathcal{F}(f)}_{\text{a differential}}$$

4. A functional is STATIONARY, if $\delta \mathcal{F} = 0$ for arbitrary infinitesimal δfs

i.e.: $\mathcal{F}(f + df) = \mathcal{F}(f)$

for any small δf.

5. Evaluation of variations

5.1 In binary products

$$\mathcal{F}(f) = \langle f | g \rangle$$

$$\begin{aligned}
\delta \mathcal{F} &= \mathcal{F}(f+\delta f) - \mathcal{F}(f) \\
&= \langle f + \delta f | g \rangle - \langle f | g \rangle \\
&= \langle \delta f | g \rangle
\end{aligned}$$

5.2 In square norms

$$\mathcal{F}(f) = \langle f | f \rangle$$

$$\begin{aligned}
\delta \mathcal{F} &= \mathcal{F}(f+\delta f) - \mathcal{F}(f) \\
&= \langle f + \delta f | f + \delta f \rangle - \langle f | f \rangle \\
&= \cancel{\langle f | f \rangle} + \langle \delta f | f \rangle + \langle f | \delta f \rangle + \mathcal{O}(2) \\
&\quad - \cancel{\langle f | f \rangle} \\
&= \langle \delta f | f \rangle + \langle f | \delta f \rangle \\
&= \langle \delta f | f \rangle + \text{c. c.}
\end{aligned}$$

complex conjugate

6. Role of + c.c. terms (iii)

If $\delta J = 0$ and

$$\delta J = \langle \delta f | g \rangle + \underbrace{\langle g | \delta f \rangle}_{\text{"c.c"}}, \qquad (*)$$

~~and~~

then $\langle \delta f | g \rangle = 0$ and $\langle g | \delta f \rangle = 0$ individually for arbitrary δf.

Proof:

Let $\delta f \to i \delta f$ (since it's arbitrary)

$$-i \langle \delta f | g \rangle + i \langle g | \delta f \rangle = 0 \qquad (1)$$

Multiplying (*) by i:

$$i \langle \delta f | g \rangle + i \langle g | \delta f \rangle = 0 \qquad (2)$$

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(2) − (1):

$$2i \underbrace{\langle \delta f | g \rangle}_{0} = 0$$

(2) + (1):

$$2i \underbrace{\langle g | \delta f \rangle}_{0} = 0$$

q.e.d